



TITLE:

Lifts of holonomy representations and the volume of a knot complement (Intelligence of Low- dimensional Topology)

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CITATION:

Goda, Hiroshi. Lifts of holonomy representations and the volume of a knot complement (Intelligence of Low-dimensional Topology). 数理解析研究所講究録 2017, 2052: 109-120

ISSUE DATE:

2017-10

URL:

<http://hdl.handle.net/2433/237117>

RIGHT:

Lifts of holonomy representations and the volume of a knot complement

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1 Introduction

A fundamental invariant of a knot is its Alexander polynomial ([1]). It has been studied for a long time from many viewpoints as a fundamental and important knot invariant. In 1990, Lin ([12]) introduced the twisted Alexander polynomial associated with a knot K and a representation $\pi_1(S^3 \setminus K) \rightarrow \mathrm{GL}(m, \mathbb{F})$ using its Seifert surface, where \mathbb{F} is a field. Subsequently, Wada ([19]) showed a method to define it using only a presentation of a group. Via its interpretations as the Reidemeister torsion by Kitano ([8]) and Kirk-Livingston ([7]), the twisted Alexander polynomial is a mathematical object which is investigated from various viewpoints now.

A 3-manifold which admits a complete Riemannian metric with sectional curvature -1 at each interior point is said to be *hyperbolic*. If a knot complement becomes a hyperbolic manifold, the knot is called a *hyperbolic knot*. It was shown by Thurston that a knot which is neither a torus knot nor a satellite knot is hyperbolic, and almost all knots are hyperbolic in the feeling. By the Mostow's rigidity theorem, which includes that the hyperbolic structure of a hyperbolic manifold is unique, we know the volume of a knot complement is an invariant of the knot.

There are several researches on estimates of the volume of a knot complement recently. In this note we consider the twisted Alexander polynomial of a hyperbolic knot associated with the representation given by the composition of the lift of the holonomy representation to $\mathrm{SL}(2, \mathbb{C})$ and the higher-dimensional, irreducible, complex representation of $\mathrm{SL}(2, \mathbb{C})$. Then we study a relationship between its asymptotic behavior and the volume of the knot.

2 Alexander polynomials

There are some methods to define the Alexander polynomial. Here we introduce the definition using a presentation of the fundamental group of a knot complement and the free differential calculus devised by Fox. Let K be a knot in the 3-sphere S^3 . Fix a Wirtinger presentation of the knot group $G(K) = \pi_1(E(K)) = \pi_1(S^3 - \mathrm{Int}N(K))$:

$$P = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle.$$

We denote by $\phi : F_n \rightarrow G(K)$ the epimorphism from the free group associated with P to $G(K)$, and by $\tilde{\phi} : \mathbb{Z}F_n \rightarrow \mathbb{Z}G(K)$ the ring homomorphism which is obtained from ϕ by extending linearly. Let $\alpha : G(K) \rightarrow H_1(E(K); \mathbb{Z}) \cong \mathbb{Z} = \langle t \rangle$ be the abelianization homomorphism. It is given by $\alpha(x_1) = \cdots \alpha(x_n) = t$ since P is a Wirtinger presentation. By extending linearly, we have a homomorphism between group rings: $\tilde{\alpha} : \mathbb{Z}G(K) \rightarrow \mathbb{Z}[t, t^{-1}]$. We denote by Φ the composed mapping $\tilde{\alpha} \circ \tilde{\phi}$, that is,

$$\Phi : \mathbb{Z}F_n \rightarrow \mathbb{Z}[t, t^{-1}].$$

The map $\frac{\partial}{\partial x_j} : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$ is the linear extension of the map defined on the elements of F_n by (1) $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$, (2) $\frac{\partial x_i^{-1}}{\partial x_j} = -\delta_{ij}x_i^{-1}$, (3) $\frac{\partial(uv)}{\partial x_j} = \frac{\partial u}{\partial x_j} + u\frac{\partial v}{\partial x_j}$. This is called the *Fox's free differential*. We obtain a matrix whose size is $(n-1) \times n$:

$$A = \left(\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \right) \in M(n-1, n; \mathbb{Z}[t, t^{-1}])$$

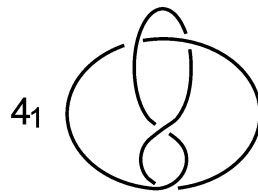
by applying the Fox's free differential to the relations r_1, \dots, r_{n-1} of the Wirtinger presentation P and composing Φ . The matrix A is called the *Alexander matrix* associated with the presentation P of the knot group $G(K)$.

We denote by A_j obtained from A by deleting the j column of A . This becomes a square matrix and we define the *Alexander polynomial* of a knot K by

$$\Delta_K(t) = \det A_j \in \mathbb{Z}[t, t^{-1}].$$

It is known that this becomes a knot invariant up to $\pm t^s$ ($s \in \mathbb{Z}$).

Example 2.1. The knot illustrated below is called the *figure eight knot* and it is known as a hyperbolic knot. The knot number is 4_1 .



Its knot group has the next presentation, which is a Wirtinger presentation:

$$G(K) = \langle x, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle.$$

We apply the Fox's free differential to the relation: $r = xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}$, then we have:

$$\begin{aligned}
 \frac{\partial}{\partial x}r &= \frac{\partial x}{\partial x} + x \frac{\partial}{\partial x}(y^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}) \\
 &= 1 + x \left(\frac{\partial y^{-1}}{\partial x} + y^{-1} \frac{\partial}{\partial x}(x^{-1}yxy^{-1}xyx^{-1}y^{-1}) \right) \\
 &= 1 + xy^{-1} \left(\frac{\partial x^{-1}}{\partial x} + x^{-1} \frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1}) \right) \tag{2.1} \\
 &= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1} \frac{\partial}{\partial x}(yxy^{-1}xyx^{-1}y^{-1}) = \dots = \\
 &= 1 - xy^{-1}x^{-1} + xy^{-1}x^{-1}y + xy^{-1}x^{-1}yxy^{-1} - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}.
 \end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y}r = -xy^{-1} + xy^{-1}x^{-1} - xy^{-1}x^{-1}yxy^{-1} + xy^{-1}x^{-1}yxy^{-1}x - xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1}.$$

Since $\alpha(x) = \alpha(y) = t$, we have:

$$\begin{aligned}
 \Phi\left(\frac{\partial r}{\partial x}\right) &= 1 - tt^{-1}t^{-1} + tt^{-1}t^{-1}t + tt^{-1}t^{-1}ttt^{-1} - tt^{-1}t^{-1}ttt^{-1}ttt^{-1} \\
 &= 1 - \frac{1}{t} + 1 + 1 - t = -\frac{1}{t} + 3 - t, \\
 \Phi\left(\frac{\partial r}{\partial y}\right) &= -1 + t^{-1} - 1 + t - 1 = \frac{1}{t} - 3 + t.
 \end{aligned}$$

Thus the Alexander matrix is the matrix of 1×2 : $\begin{pmatrix} -\frac{1}{t} + 3 - t & \frac{1}{t} - 3 + t \end{pmatrix}$, and the Alexander polynomial of the figure eight knot K is:

$$\Delta_K(t) = \det \left(-\frac{1}{t} + 3 - t \right) = -\frac{1}{t} + 3 - t$$

(up to $\pm t^s$ ($s \in \mathbb{Z}$)).

Originally x and y are different generators in the knot group, but they are sent to the same element t by the map α . It makes the calculation easy while this process might reduce some information included in knot groups. The twisted Alexander polynomial, which is introduced in the following section, improves this point.

3 Twisted Alexander polynomials

We use the same notations as in the previous sections. Let $\rho : G(K) \rightarrow \text{SL}(m, \mathbb{C})$ be a representation of a knot group $G(K)$. This map induces naturally the map between group rings:

$$\tilde{\rho} : \mathbb{Z}G(K) \rightarrow M(m; \mathbb{C}),$$

moreover by taking the tensor product with the map $\tilde{\alpha}$ induced in the previous section, we have:

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbb{Z}G(K) \rightarrow M(m, \mathbb{C}[t, t^{-1}]).$$

Set Φ :

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbb{Z}F_n \rightarrow M(m; \mathbb{C}[t, t^{-1}])$$

by composing $\tilde{\phi}$ defined in the previous section. Suppose A_ρ is the $(n-1) \times n$ matrix whose the (i, j) element is the $m \times m$ matrix:

$$\Phi\left(\frac{\partial r_i}{\partial x_j}\right) \in M((n-1)m \times nm; \mathbb{C}[t, t^{-1}]).$$

This matrix is called the *twisted Alexander matrix associated with ρ* . In order to make a square matrix, we delete from A_ρ ‘one column’ corresponding to a generator x_k in the presentation P , so that we have a $(n-1)m \times (n-1)m$ matrix, which is denoted by $A_{\rho,k}$. We define the *twisted Alexander polynomial* as:

$$\Delta_{K,\rho}(t) = \frac{\det A_{\rho,k}}{\det \Phi(x_k - 1)}.$$

Here we assume $\det \Phi(x_k - 1) \neq 0$.

Wada proved the following theorem in [19].

Theorem 3.1 ([19]). *Let K be a knot and $G(K)$ the knot group. Suppose ρ is a representation of $G(K)$. The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is an invariant for the pair $(G(K), \rho)$ up to $\pm t^s$ ($s \in \mathbb{Z}$).*

Example 3.2. Let K be the figure eight knot, then the knot group $G(K)$ has the following presentation as in Example 2.1:

$$G(K) = \langle x, y \mid xy^{-1}x^{-1}yxy^{-1}xyx^{-1}y^{-1} \rangle. \quad (3.1)$$

Define

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ \frac{-1+\sqrt{-3}}{2} & 1 \end{pmatrix}. \quad (3.2)$$

Then we can confirm that ρ becomes a representation from $G(K)$ to $\mathrm{SL}(2, \mathbb{C})$. Set $\rho(x) = X$, $\rho(y) = Y$, then we have : $\Phi\left(\frac{\partial r}{\partial x}\right) =$

$$I - \frac{1}{t}XY^{-1}X^{-1} + XY^{-1}X^{-1}Y + XY^{-1}X^{-1}YXY^{-1} - tXY^{-1}X^{-1}YXY^{-1}XYX^{-1},$$

where I is the identity matrix of size 2×2 . Note that this can be obtained from (2.1) by changing $x \rightarrow X$, $y \rightarrow Y$ and $1 \rightarrow I$ with t to the appropriate power. Calculate these matrices, then we have:

$$\Delta_{K,\rho}(t) = \frac{\det \Phi\left(\frac{\partial r}{\partial x}\right)}{\det \Phi(y - 1)} = \frac{1/t^2(t-1)^2(t^2 - 4t + 1)}{(t-1)^2} \doteq t^2 - 4t + 1.$$

This seems to make up for the lack of the information caused by going through the map α . However the twisted Alexander polynomial depends on not only $G(K)$ but also ρ , so it might not be called a knot invariant, namely, it is hard to use for distinguishing two given knots. Furthermore, it is not easy to find a representation of a knot group in general. Therefore the thinkable ways to apply might be (1) to find a property of a knot satisfied for any representation or (2) to consider the restricted representation. I think an example of the former case is to determine a non-fibered knot by using any unimodular representation, i.e., we gave the theorem in [6] which states that the twisted Alexander polynomials of fibered knots are monic for any unimodular representation. See [3, 15] for the researches which followed this theorem. In the following sections, we will consider the twisted Alexander polynomial associated with the holonomy representation of a hyperbolic knot, which corresponds to the case (2) above.

For the details of basic notations and conceptions on the twisted Alexander polynomial, see [10]. See [4, 9] for its recent researches.

4 On hyperbolic knots

We refer [11, 18] for the former half in this section.

We regard the upper half space model \mathbb{H}^3 as a subspace of the quaternion field and set

$$\mathbb{H}^3 = \{(x + yi) + tj \in \mathbb{C} + \mathbb{R}j \mid t > 0\}$$

where $1, i, j$ are the part of basis, $i = \sqrt{-1}$, and we suppose $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. We give the metric

$$ds^2 = \frac{1}{t^2}(dx^2 + dy^2 + dt^2)$$

to \mathbb{H}^3 then we call this \mathbb{H}^3 *the 3-dimensional hyperbolic space*. It is known that the orientation preserving isometric transformation group of \mathbb{H}^3 is:

$$\mathrm{PSL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\} / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Here the action on \mathbb{H}^3 of $\mathrm{PSL}(2, \mathbb{C})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} w = (aw + b)(cw + d)^{-1} \quad (w \in \mathbb{H}^3).$$

We calculate the right-hand side as elements of the quaternion field. The isometric transformation of \mathbb{H}^3 is a conformal mapping, and the action of $\mathrm{PSL}(2, \mathbb{C})$ on \mathbb{H}^3 is transitive. Moreover its stabilizer of a point is $\mathrm{PSU}(2, \mathbb{C}) (\cong \mathrm{SO}(3))$, that is, if $f(p)$ and the mapping between tangent spaces $T_p\mathbb{H}^3 \rightarrow T_{f(p)}\mathbb{H}^3$ are given for $f \in \mathrm{PSL}(2, \mathbb{C})$ and a point $p \in \mathbb{H}^3$, then f may be

determined uniquely. Thus, if the image of the neighborhood of a point by the isometric transformation is given, then one can extend the mapping to the whole space \mathbb{H}^3 uniquely. Further, there exists uniquely the transformation $f \in \text{PSL}(2, \mathbb{C})$ such that f transforms any 3 points $p_1, p_2, p_3 \in \partial\mathbb{H}^3$ into any 3 points $p'_1, p'_2, p'_3 \in \partial\mathbb{H}^3$.

Let M be a 3-dimensional differentiable manifold. If M has a local coordinate such that a neighborhood of each point is homeomorphic to an open set in \mathbb{H}^3 and the coordinate transformation can be written in an element of $\text{PSL}(2, \mathbb{C})$, we call M a *hyperbolic* 3-manifold. This is the same concept defined in Section 1. For a simply-connected hyperbolic 3-manifold M' , we may define the developing map from M' to \mathbb{H}^3 as follows. Give a local coordinate of a neighborhood of the base point in M' . For any point p we take a path from the base point to p and a sequence of local coordinates along the path. We may have the image of p by the developing map by determining the image of the coordinate functions corresponding to the sequence in order. (It does not depend on the way to take a path.) Let γ be an element of the fundamental group of a hyperbolic 3-manifold M and $\tilde{\gamma}$ a lift of γ to a universal covering \tilde{M} of M . We call the homomorphism $\rho : \pi_1(M) \rightarrow \text{PSL}(2, \mathbb{C})$ the *holonomy representation* of M if $\rho(\gamma)$ is the element ($\in \text{PSL}(2, \mathbb{C})$) which maps the image by the developing map of the neighborhood of the base point of $\tilde{\gamma}$ to that of the neighborhood of the end point of $\tilde{\gamma}$. Let Γ be the image $\rho(\pi_1(M))$ for the holonomy representation ρ of a complete hyperbolic 3-manifold M , then Γ acts \mathbb{H}^3 naturally and M is homeomorphic to \mathbb{H}^3/Γ . Therefore the classification of complete hyperbolic 3-manifolds is equivalent essentially to that of a kind of discrete subgroups of $\text{PSL}(2, \mathbb{C})$, so we may think the geometrical information of a complete hyperbolic 3-manifold is included in Γ . It is shown by Thurston that a holonomy representation can be lift to $\text{SL}(2, \mathbb{C})$ representation, and in [2] it is proved that the lift has a one-to-one correspondence to the spin structure of M . Let η be a spin structure of M . Then we have the following homomorphism:

$$\text{Hol}_{(M, \eta)} : \pi_1(M, \eta) \rightarrow \text{SL}(2, \mathbb{C}).$$

If a submanifold of a hyperbolic 3-manifold is homeomorphic to the direct product of the 2-dimensional torus and the half-line, the submanifold is called a *cuspidal*. A 3-manifold M with a cusp is non-compact, and we obtain from M a compact 3-manifold whose boundary is a torus by getting rid of a neighborhood of the cusp. It is known that a complete hyperbolic 3-manifold with finite volume is a closed 3-manifold or a 3-manifold with cusps. In particular, a knot K (L resp.) is said to be *hyperbolic* if $S^3 - K$ ($S^3 - L$ resp.) admits the structure of the hyperbolic 3-manifold with a cusp (cusps resp.). A knot which is neither a torus knot nor a satellite knot is hyperbolic.

In the case of a knot in S^3 , we let A_1, \dots, A_n be the images of generators a_1, \dots, a_n of a Wirtinger presentation of $G(K)$ by the holonomy representation ρ , then their lifts to $\text{SL}(2, \mathbb{C})$ are A_1, \dots, A_n or $-A_1, \dots, -A_n$ (Corollary 2.3 in [14]). We denote by $\rho^\pm(a_i) = \pm A_i (\in$

$\mathrm{SL}(2, \mathbb{C})$) for the lifts of the holonomy representation $\rho(a_i) = A_i (\in \mathrm{PSL}(2, \mathbb{C}))$.

5 Irreducible $\mathrm{SL}(m, \mathbb{C})$ -representations of $\mathrm{SL}(2, \mathbb{C})$

We review irreducible representations of $\mathrm{SL}(2, \mathbb{C})$ briefly. The vector space \mathbb{C} has the standard action of $\mathrm{SL}(2, \mathbb{C})$. It is known that the symmetric product $\mathrm{Sym}^{m-1}(\mathbb{C}^2)$ and the induced action by $\mathrm{SL}(2, \mathbb{C})$ give an m -dimensional representation of $\mathrm{SL}(2, \mathbb{C})$. We can identify $\mathrm{Sym}^{m-1}(\mathbb{C}^2)$ with the vector space of homogeneous polynomials on \mathbb{C}^2 with degree $m - 1$, namely,

$$V_m = \mathrm{span}_{\mathbb{C}} \langle x^{m-1}, x^{m-2}y, \dots, xy^{m-2}, y^{m-1} \rangle.$$

The action of $A \in \mathrm{SL}(2, \mathbb{C})$ is expressed as

$$A \cdot p \begin{pmatrix} x \\ y \end{pmatrix} = p \left(A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

where $p \begin{pmatrix} x \\ y \end{pmatrix}$ is a homogeneous polynomial and the variables in the right-hand side are determined by the action of A^{-1} on the column vector as a matrix multiplication. We denote by (V_m, σ_m) the representation given by the above action of $\mathrm{SL}(2, \mathbb{C})$ where σ_m denotes the homomorphism from $\mathrm{SL}(2, \mathbb{C})$ into $\mathrm{GL}(V_m)$. It is known that (1) each representation (V_m, σ_m) turns into an irreducible $\mathrm{SL}(m, \mathbb{C})$ -representation and (2) every irreducible m -dimensional representation of $\mathrm{SL}(2, \mathbb{C})$ is equivalent to (V_m, σ_m) .

Let M be a complete hyperbolic 3-manifold and $\mathrm{Hol}_{(M, \eta)}$ the homomorphism defined in Section 4. By composing $\mathrm{Hol}_{(M, \eta)}$ and σ_m , we have the representation:

$$\rho_m : \pi_1(M) \rightarrow \mathrm{SL}(m, \mathbb{C}).$$

Example 5.1. It is known that the map given by (3.2) in Example 3.2 is the holonomy representation of the figure eight knot K . To avoid the reduplication, let a and b be generators of $G(K)$ instead of x and y in the group presentation (3.1) and set:

$$\rho(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ \frac{-1+\sqrt{-3}}{2} & 1 \end{pmatrix}.$$

Note that these are the elements in $\mathrm{SL}(2, \mathbb{C})$. Since $(x - y)^2 = x^2 - 2xy + y^2$, $(x - y)y = xy - y^2$, $y^2 = y^2$, we have the next matrix by taking the coefficients:

$$\rho_3(a) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^T.$$

By setting $u = \frac{-1 + \sqrt{-3}}{2}$ and calculating similarly, we obtain:

$$\rho_3(b) = \begin{pmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ u^2 & 2u & 1 \end{pmatrix}^T, \quad \rho_4(a) = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T, \quad \rho_4(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & 1 & 0 & 0 \\ u^2 & 2u & 1 & 0 \\ u^3 & 3u^2 & 3u & 1 \end{pmatrix}^T.$$

Here $(\cdot)^T$ means the transposed matrix.

6 Main Theorem and the outline of the proof

Let K be a hyperbolic knot, and ρ_m the $\mathrm{SL}(m, \mathbb{C})$ -representation which is obtained from the holonomy representation of $G(K)$ by the method described in Sections 4 and 5. Set:

$$\mathcal{A}_{K,2k}(t) = \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}; \quad \mathcal{A}_{K,2k+1}(t) = \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}. \quad (6.1)$$

Our main result is the following:

Theorem 6.1 ([5]).

$$\lim_{k \rightarrow \infty} \frac{\log |\mathcal{A}_{K,2k}(1)|}{(2k)^2} = \lim_{k \rightarrow \infty} \frac{\log |\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2} = \frac{\mathrm{Vol}(K)}{4\pi}.$$

As in (6.1), \mathcal{A}_{K,ρ_m} is defined by dividing the principal part, but it is inessential, especially in the case of m even. We may describe as follows if there is no corrections:

- $\lim_{k \rightarrow \infty} \frac{\log |\Delta_{K,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi};$
- $\lim_{k \rightarrow \infty} \frac{1}{(2k+1)^2} \left(\log \left(\lim_{t \rightarrow 1} \left| \frac{\Delta_{K,2k+1}(t)}{t-1} \right| \right) \right) = \frac{\mathrm{Vol}(K)}{4\pi}.$

In the next section, we give sample calculations of the figure eight knot. As shown there the volume of a knot complement can be approximated using a kind of a combinatorial method. The crucial points are the results of Müller: one of them states the analytic torsion and the Reidemeister torsion are the same essentially for unimodular representations ([16]) and the other gives the volume formula using the analytic torsion ([17]) for a closed complete hyperbolic 3-manifold. Thus, combining them, we are able to have a volume formula for a closed complete hyperbolic 3-manifold using the Reidemeister torsion. Applying the Thurston's hyperbolic Dehn surgery theorem to these Müller's works, Menal-Ferrer and Porti gave a volume formula for a complete hyperbolic 3-manifold with cusps in [14] (see Theorem 6.4), so we have only to make clear the relation between the Reidemeister torsion and the twisted Alexander polynomial.

Let us review some results of Menal-Ferrer and Porti. Let M be an oriented complete hyperbolic 3-manifold whose boundary is one torus cusp, i.e., we will consider M with $\partial \overline{M} = T^2$.

Proposition 6.2 ([13]). (1) If m is even, then $\dim_{\mathbb{C}} H_i(M; \rho_m) = 0$ for any i .

(2) If m is odd, then $\dim_{\mathbb{C}} H_0(M; \rho_m) = 0$ and $\dim_{\mathbb{C}} H_i(M; \rho_m) = 1$ for $i = 1, 2$.

Proposition 6.3 ([14]). Suppose m is odd and let $G < \pi_1(M)$ be some fixed realization of the fundamental group of T as a subgroup of $\pi_1(M)$. Choose a non-trivial cycle $\theta \in H_1(T; \mathbb{Z})$, and a non-trivial vector $v \in V_m$ fixed by $\rho_m(G)$. If $i : T \rightarrow M$ denotes the inclusion, then the following assertions hold.

(1) A basis for $H_1(M; \rho_m)$ is given by $i_*([v \otimes \theta])$.

(2) Let $[T] \in H_2(T; \mathbb{Z})$ be a fundamental class of T . A basis for $H_2(M; \rho_m)$ is given by $i_*([v \otimes T])$.

Using the above notations, we set:

$$\mathcal{T}_{2k+1}(M) = \frac{\text{Tor}(M; \rho_{2k+1}; \theta)}{\text{Tor}(M; \rho_3; \theta)};$$

$$\mathcal{T}_{2k}(M) = \frac{\text{Tor}(M; \rho_{2k})}{\text{Tor}(M; \rho_2)}.$$

Here $\text{Tor}(\cdot)$ means the Reidemeister torsion.

Theorem 6.4 ([14]).

$$\lim_{k \rightarrow \infty} \frac{\log |\mathcal{T}_{2k+1}(M)|}{(2k+1)^2} = \lim_{k \rightarrow \infty} \frac{\log |\mathcal{T}_{2k}(M)|}{(2k)^2} = \frac{\text{Vol}(M)}{4\pi}.$$

As in Proposition 6.2 (1), the twisted homology vanishes in the case that m is even. In such a case the corresponding chain complex is said to be *acyclic* and it is easy relatively to discuss the Reidemeister torsion. Let M be the complement $E(K)$ of a knot K . It is proved by Kitano ([8]) that the Reidemeister torsion can be obtained from the twisted Alexander polynomial by evaluating $t = 1$ in this case, that is,

$$\text{Tor}(M; \rho_{2k}) = \Delta_{K, \rho_{2k}}(1).$$

Thus we get the even case of our main result via Theorem 6.4

The representation obtained from the adjoint action of the $\text{SL}(2, \mathbb{C})$ -representation of a fundamental group is the same as ρ_3 in our setting essentially. The next proposition is a generalization of the Yamaguchi's theorem ([20, 21]) which treats the adjoint action of the $\text{SL}(2, \mathbb{C})$ -representation of a fundamental group. We restrict the base θ in Proposition 6.3 to a longitude λ and handle it well, so that we have this proposition:

Proposition 6.5 ([5]). Let λ be a longitude of a knot K and M the complement of K , then the following equation holds:

$$|\text{Tor}(M; \rho_{2k+1}; \lambda)| = \lim_{t \rightarrow 1} \frac{|\Delta_{K, \rho_{2k+1}}(t)|}{t - 1}.$$

The odd case in our main result follows from the proposition.

7 Some calculations

Here we give some calculations on the figure eight knot K . It is known that the volume of the complement of K is equal to $2.0298832 \dots$.

We use the lifts $\rho^+(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\rho^+(b) = \begin{pmatrix} 1 & 0 \\ \frac{-1+\sqrt{-3}}{2} & 1 \end{pmatrix}$, stated in Example 5.1, and we proceed the calculation in Example 3.2, then we have:

$$\Delta_{K,\rho_2^+}(t) = \frac{1}{t^2}(t^2 - 4t + 1), \Delta_{K,\rho_3^+}(t) = -\frac{1}{t^3}(t-1)(t^2 - 5t + 1), \Delta_{K,\rho_4^+}(t) = \frac{1}{t^4}(t^2 - 4t + 1)^2.$$

In the same way, we can have:

$$\Delta_{K,\rho_5^+}(t) = -\frac{1}{t^5}(t-1)(t^4 - 9t^3 + 44t^2 - 9t + 1).$$

We denote by $\mathcal{A}_{K,m}^+$ the corresponding $\mathcal{A}_{K,m}$ with ρ^+ , so we obtain:

$$\begin{aligned} \frac{4\pi \log |\mathcal{A}_{K,4}^+(t)|}{4^2} &= \frac{\pi \log |t^2 - 4t + 1|}{4} \xrightarrow{t=1} \frac{\pi \log 2}{4} \approx 0.544397 \dots; \\ \frac{4\pi \log |\mathcal{A}_{K,5}^+(t)|}{5^2} &= \frac{\pi \log \left| \frac{t^4 - 9t^3 + 44t^2 - 9t + 1}{t^2 - 5t + 1} \right|}{5^2} \xrightarrow{t=1} \frac{4\pi \log \frac{28}{3}}{5^2} \approx 1.12273 \dots. \end{aligned}$$

The following is the results using by a computer. The symbol $\mathcal{A}_{K,m}^-$ corresponds to the lift of the holonomy representation of K :

$$\rho^-(a) = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \rho^-(b) = -\begin{pmatrix} 1 & 0 \\ \frac{-1+\sqrt{-3}}{2} & 1 \end{pmatrix}.$$

Note that $\mathcal{A}_{K,m}^+(t) = \mathcal{A}_{K,m}^-(t)$ when m is odd. Mr. Tetsuya Takahashi helped me to calculate these and we used the softwares Wolfram Mathematica and MathWorks Matlab. It took about 4 ~ 5 hours to compute in the degree 33 case.

$m(\text{even})$	$\frac{4\pi \log \mathcal{A}_{K,m}^+(1) }{m^2}$	$\frac{4\pi \log \mathcal{A}_{K,m}^-(1) }{m^2}$	$m(\text{odd})$	$\frac{4\pi \log \mathcal{A}_{K,m}(1) }{m^2}$
4	0.54439...	1.40724...	5	1.12273...
8	1.66441...	1.84668...	9	1.76436...
12	1.86678...	1.94781...	13	1.90158...
16	1.93822...	1.98381...	17	1.95494...
20	1.97121...	2.00039...	21	1.98076...
24	1.98914...	2.00940...	25	1.99522...
28	1.99994...	2.01483...	29	2.00412...
32	2.00696...	2.01836...	33	2.00999...

Acknowledgements

This work was supported by JSPS KAKENHI Grant Numbers JP15K04868.

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